

ZERO-CYCLES AND RATIONAL POINTS ON VARIETIES DEFINED BY ABELIAN NORMIC EQUATIONS

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ABSTRACT. We study the Hasse principle for 0-cycles on proper smooth varieties X defined by Abelian normic equations $N_{K/k}(\vec{x}) = P(t)$. We aim to prove the exactness of a sequence of local-global type related to the Brauer group of X , our main result reduces this long raised question to explicit computation of ramifications of (finitely many) elements of a certain Brauer group. The main theorem also generalizes some existing results on this topic.

A similar statement for rational points is also valid assuming Schinzel's hypothesis.

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1. INTRODUCTION

Let k be a number field and Ω_k be the set of all places of k . Consider an algebraic k -variety X assumed proper smooth and geometrically integral, denote by $Br(X) = H_{\text{ét}}^2(X, \mathbb{G}_m)$ the cohomological Brauer group of X . We write simply $X_v = X \times_k k_v$ for each place $v \in \Omega_k$. We define the *modified Chow group* of 0-cycles $CH'_0(X_v)$ to be either the usual Chow group if v is a non-archimedean place, or $Coker[CH_0(X_v \times_{k_v} \bar{k}_v) \xrightarrow{N_{\bar{k}_v/k_v}} CH_0(X_v)]$ if v is an archimedean place.

As a variant of Manin's pairing, one can define by taking the sum of local pairings $\langle \cdot, \cdot \rangle_v$

$$Br(X) \times \prod_{v \in \Omega_k} CH'_0(X_v) \rightarrow \mathbb{Q}/\mathbb{Z}$$

$$(b, \{z_v\}_v) \mapsto \sum_{v \in \Omega_k} inv_v(\langle b, z_v \rangle_v),$$

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where $\text{inv}_v : Br(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the local invariant, *cf.* [CT95]. By class field theory, global 0-cycles are annihilated under the pairing, we get a complex

$$(E) \quad \varprojlim_n CH_0(X)/n \rightarrow \prod_{v \in \Omega_k} \varprojlim_n CH'_0(X_v)/n \rightarrow \text{Hom}(Br(X), \mathbb{Q}/\mathbb{Z}),$$

where $/n$ denote the cokernel of the multiplication by n . The exactness of (E) means roughly that the failure of the local-global principle for 0-cycles can be controlled by the Brauer group $Br(X)$. It is believed to be exact for all smooth proper geometrically integral varieties after the work of Colliot-Thélène, Kato, Sansuc, Saito, *cf.* [CTS81] [KS86] [CT95]. However, this is far from being completely proved even though the exactness has been found out to be valid for more and more families of varieties.

In this paper, we focus on varieties defined by *Abelian* normic equations

$$N_{K/k}(\vec{x}) = P(t),$$

where K/k is a finite *Abelian* extension with $P(t) \in k[t]$ a polynomial, and where \vec{x} represents an element in K written as a k -linear combination of a chosen base of the k -vector space K . When K/k is a cyclic extension, the exactness of (a variant of) the sequence (E) was discussed in a more general setting in [CTSSD98, §4], where the parallel question for rational points was discussed as well. In his recent paper [Wei], Wei discussed the Hasse principle for 0-cycles of degree 1 in some scattered cases which are not covered by [CTSSD98]. Other cases are still open, even in the simplest non-cyclic case where K/k is a biquadratic extension, *cf.* [CTSSD98, Remark 1.5]. Trying to prove the exactness of (E) for varieties defined by such equations, to lead to a more systematic discussion we propose to divide the question into two steps:

- Q1. general statement for the exactness of (E) with minimum assumptions on elements in certain Brauer groups;
- Q2. explicit computation for Brauer elements and verification of the assumptions appeared in Q1.

Our main result is Theorem 2.1, which gives an answer to Q1 with an assumption (Br) on Brauer groups. As examples, smooth proper models of an arbitrary Abelian normic equation satisfy geometrical assumptions of the main theorem. Hence we reduce the exactness of (E) to Q2. Theorem 2.1 is also a generalization of several existing results, *cf.* §5.1. Combining the work of Wei with our key ingredient Proposition 3.5, we also obtain the exactness of (E) for the examples discussed by Wei, *cf.* §5.2.

Concerning the parallel question for rational points, the Brauer-Manin obstruction is expected to be the only obstruction for rational points on rationally connected varieties (for example, varieties defined by normic equations), *cf.* [CTS77]. In the last section §6, we will state and explain a similar result for rational points on varieties defined by Abelian normic equations admitting Schinzel's hypothesis.

2. STATEMENT OF THE MAIN RESULT

Let X be a smooth proper geometrically integral variety defined over a number field k . Fix an integer δ . If the existence of a family $\{z_v\}_{v \in \Omega_k}$ of local 0-cycles of degree δ orthogonal to $Br(X)$ implies the existence of a global 0-cycle of degree δ on X , then we say that *the Brauer-Manin obstruction is the only obstruction to the Hasse principle* for 0-cycles of degree δ on X . We say that *the Brauer-Manin*

obstruction is the only obstruction to weak approximation for 0-cycles of degree δ on X if the following statement is satisfied

For any positive integer N and any finite set S of places of k , given an arbitrary family of local 0-cycles $\{z_v\}_{v \in \Omega_{k(\theta)}}$ of degree δ orthogonal to $Br(X)$, then there exists a global 0-cycle $z = z_{S,N}$ of degree δ on X such that z and z_v have the same image in $CH_0(X_v)/N$ for all $v \in S$.

These definitions date back to [CTSD94, page 69].

Let $\pi : X \rightarrow \mathbb{P}^1$ be a fibration (dominant morphism whose generic fiber is geometrically integral) defined over a number field k . Suppose that the variety X is smooth projective and geometrically integral. We denote by X_η the generic fiber of π , and we write $X_{\bar{\eta}} = X_\eta \times_{k(t)} \bar{k}(t)$. Suppose that $Pic(X_{\bar{\eta}})$ is torsion-free and $Br(X_{\bar{\eta}})$ is finite, then $Br(X_\eta)/Br(k(t))$ is a finite group. Let $\Lambda \subset Br(X_\eta)$ be a finite subset generating $Br(X_\eta)$ modulo $Br(k(t))$. Let F be an integral 1-codimensional closed subvariety of X . Consider the residue map $\partial_F : Br(k(X)) \rightarrow H^1(k(F), \mathbb{Q}/\mathbb{Z})$, if there exists an element $b \in \Lambda \subset Br(X_\eta)$ such that $\partial_F(b) \neq 0$ then F is contained in a certain closed fiber X_m of π . Let $\{m_r; 1 \leq r \leq l\}$ be a set of closed points of \mathbb{P}^1 . We denote by X_r the fiber X_{m_r} and we write $Z = \bigsqcup_r X_r$ and $Y = X \setminus Z$.

Theorem 2.1. *With the above notation, let $\pi : X \rightarrow \mathbb{P}^1$ be a fibration satisfying*

- (Ab-sp) *all closed fibers are Abelian-split;*
- (gen) *$Pic(X_{\bar{\eta}})$ is torsion-free and $Br(X_{\bar{\eta}})$ is finite;*
- (Br) *there exists a finite subset $\Lambda \subset Br(X_\eta)$ generating $Br(X_\eta)$ modulo $Br(k(t))$ such that $\Lambda \subset Br(Y)$ where $Y \subset X$ is an open with complement $X \setminus Y$ a disjoint union of closed split fibers.*

Let $Hil \subset \mathbb{P}^1$ be a generalized Hilbertian subset of \mathbb{P}^1 , we suppose respectively that for all $\theta \in Hil$

- (1) *the Brauer-Manin obstruction is the only obstruction to the Hasse principle for rational points (or 0-cycles of degree 1) on X_θ ;*
- (2) *the Brauer-Manin obstruction is the only obstruction to weak approximation for rational points (or 0-cycles of degree 1) on X_θ ;*
- (3) *we suppose (2), moreover, the induced map $CH_0(X_v) \rightarrow CH_0(\mathbb{P}_v^1) \simeq \mathbb{Z}$ is assumed injective for almost all places v .*

Then for every integer δ , we have respectively

- (1) *the Brauer-Manin obstruction is the only obstruction to the Hasse principle for 0-cycles of degree δ on X ;*
- (2) *the Brauer-Manin obstruction is the only obstruction to weak approximation for 0-cycles of degree δ on X ;*
- (3) *the Brauer-Manin obstruction is the only obstruction to strong approximation for 0-cycles of degree δ on X , and the sequence (E) is exact for X .*

Remark 2.2. The induced map $CH_0(X_v) \rightarrow CH_0(\mathbb{P}_v^1) \simeq \mathbb{Z}$ is the degree map, it is injective for almost all v if X_η is assumed rationally connected thanks to a theorem of Kollár/Szabó [KS03, Thm. 5]. It is also well-known that if X_η is rationally connected, the hypothesis (gen) is automatically satisfied, cf. [Gro68, II Cor. 3.4], the proof of [CTR85, Prop. 2.11], and [Deb01, Cor. 4.18].

Smooth proper models of the equation $N_{K/k}(\vec{x}) = P(t)$ with K/k an Abelian extension satisfy the hypothesis (Ab-sp), cf. [Liab, Lem. 3.4]. These models are

fibered over \mathbb{P}^1 by the parameter t , almost all fibers are smooth compactifications of torsors under algebraic tori, the arithmetic hypothesis (1) and (2) are satisfied.

However, the verification of the hypothesis (Br) on the Brauer group does not seem obvious. This is caused by the difficulty of describing explicitly the representatives in Λ . Even for a specific Abelian extension K/k , it is not clear whether there exists a smooth proper model for the normic equation verifying the hypothesis (Br). Concerning this hypothesis, we have a more detailed discussion in §5.

3. PRELIMINARIES TO THE PROOF

At first, we state some preliminaries and give references of each statement. In the subsection §3.4, we prove a variant of Hilbert's irreducibility theorem, which is one of the main ingredients of this work.

3.1. Formal lemma.

Lemma 3.1 ([CTSSD98, Lem. 4.5], original version [Har94, Cor. 2.6.1]). *Let X be a smooth proper geometrically integral variety defined over a number field k . Let U be a non-empty open subset of X and $\{A_1, \dots, A_n\} \subset \text{Br}(U) \subset \text{Br}(k(X))$. We denote by B the intersection of $\text{Br}(X)$ and the subgroup generated by the A_i 's in $\text{Br}(k(X))$.*

Suppose that for every $v \in \Omega_k$, there exists a 0-cycle z_v on X_v of degree δ supported in U_v such that the family $\{z_v\}_{v \in \Omega_k}$ is orthogonal to B .

Then, for all finite set S of places of k , there exists a finite set S' of places of k containing S , and for each $v \in S'$ there exists a 0-cycle z'_v on U_v of degree δ such that

$$\sum_{v \in S'} \text{inv}_v(\langle A_i, z'_v \rangle_v) = 0$$

and moreover $z'_v = z_v$ for all $v \in S$.

3.2. Moving lemmas for 0-cycles. We say that a 0-cycle $z = \sum n_P P$ is *separable* if each non-zero integer n_P equals either 1 or -1 . Let k be a topological field (of characteristic 0) and \bar{k} be its fixed algebraic closure. Let z' be an effective 0-cycle of degree $d > 0$ on a k -variety V , we express it as a sum of closed points $z' = \sum P'_i$ (not necessarily separable, the closed points P'_i may be equal for different i). We say that an effective 0-cycle $z = \sum P_i$ of degree d is *sufficiently close* to z' if (after a permutation of the indices) we have $k(P_i) = k(P'_i)$ and P_i is sufficiently close to P'_i in the topological space $V(k(P'_i))$.

Lemma 3.2 ([CT05, §3]). *Let X be a integral regular variety defined over a perfect field k , and U be a non-empty open subset of X . Then every 0-cycle z of X is rationally equivalent on X to a 0-cycle z' supported in U .*

Lemma 3.3 ([CTSD94, page 89], [CTSSD98, page 19]). *Let $\pi : X \rightarrow \mathbb{P}^1$ be a fibration defined over \mathbb{R} , \mathbb{C} , or a p -adic field. Suppose that X is smooth integral. Let D be a finite set of closed points of \mathbb{P}^1 , and X_0 be a non-empty Zariski open subset of X .*

Then for every effective 0-cycle $z \neq 0$ supported in X_0 , there exists a separable effective 0-cycle z' supported in X_0 such that z' is sufficiently close to z and such that $\pi_(z')$ is separable and supported outside D . The 0-cycles $\pi_*(z)$ and $\pi_*(z')$ are rationally equivalent on \mathbb{P}^1 .*

3.3. Comparison of Brauer groups. The following proposition was stated originally for rational points θ , but the whole proof works for closed points.

Proposition 3.4 ([Har94, Thm. 3.5.1], [Har97, Thm. 2.3.1]). *Let $X \rightarrow \mathbb{P}^1$ be a fibration defined over a number field k . Suppose that $\text{Pic}(X_{\bar{\eta}})$ is torsion-free and $\text{Br}(X_{\bar{\eta}})$ is finite.*

Then there exists a generalized Hilbertian subset $\text{Hil} \subset \mathbb{P}^1$ such that for all $\theta \in \text{Hil}$, the specialization

$$sp_{\theta} : \frac{\text{Br}(X_{\eta})}{\text{Br}(k(t))} \rightarrow \frac{\text{Br}(X_{\theta})}{\text{Br}(k(\theta))}$$

is an isomorphism of finite Abelian groups.

3.4. Hilbert's irreducibility theorem. The following proposition is a variant of Hilbert's irreducibility theorem. This is a crucial step in the proof of the main theorem. In order to prove the proposition, we follow the strategy of Ekedahl [Eke90], of which more detailed arguments were given by Harari [Har94]. We combine the method of Colliot-Thélène [CT00] to deal with 0-cycles on higher genus curves.

Proposition 3.5. *Let k be a number field and Hil be a generalized Hilbertian subset of \mathbb{P}^1_k . Let P_i ($i = 1, \dots, n$) be closed points of $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ of residue field k_i .*

Let S be a finite set of places of k and z_v be an effective separable 0-cycle of degree $d > \sum_{i=1}^n [k_i : k] - 2$ on \mathbb{P}^1_v supported in $\mathbb{A}^1 \setminus \bigcup_{i=1}^n P_i$ for each $v \in S$.

Then, given a finite non-trivial extension F of k , there exists

- *an infinite set I of places of k ,*
- *a closed point $\theta \in \mathbb{A}^1$ of degree d (defined by an irreducible polynomial $f \in k[t]$),*
- *a place $w_i \in \Omega_{k_i} \setminus S \otimes_k k_i$ for each i ,*

satisfying the following conditions

- (1) *each place in I splits completely in F ,*
- (2) *$\theta \in \text{Hil}$,*
- (3) *as a 0-cycle, θ is sufficiently close to z_v for all $v \in S$,*
- (4) *as a $k(\theta)$ -point of \mathbb{A}^1 , θ is an $S \cup I$ -integer,*
- (5) *for each i , we have $w_i(f(P_i)) = 1$ and $w(f(P_i)) = 0$ for all places $w \in \Omega_{k_i} \setminus (S \cup I) \otimes_k k_i$ different from w_i .*

Proof. The generalized Hilbertian subset $\text{Hil} \subset U \subset \mathbb{P}^1$ is defined by a finite morphism $Z \rightarrow \mathbb{P}^1$ étale over U , where Z an integral k -variety. Take K' to be the Galois closure of the extension $k(Z)/k(\mathbb{P}^1)$ of function fields. Let Z' be the normal curve having K' as its function field, then the composite of finite morphisms $Z' \rightarrow Z \rightarrow \mathbb{P}^1$ defines a generalized Hilbertian Hil' contained in Hil . We are going to find a closed point $\theta \in \text{Hil}'$.

Let k' be the algebraic closure of k in K' . By shrinking U if necessary, we may assume that $U' = Z' \times_{\mathbb{P}^1} U$ is étale surjective over U , the cover $U' \rightarrow U$ is Galois of group $G = \text{Gal}(K'/k(t))$, the open U is contained in \mathbb{A}^1 , and moreover the k -morphism $U' \rightarrow U$ factorizes through $U_{k'} \rightarrow U$ by a k' -morphism, cf. [Liaa, Lem. 1.3]. Then $U' \rightarrow U_{k'}$ is a Galois cover of group $H = \text{Gal}(K'/k'(t))$. By enlarging S if necessary, we may assume that all these extend to smooth integral models $\mathcal{U}' \rightarrow \mathcal{U}_{O_{k'}, S'} \rightarrow \mathcal{U}$ over $O_{k, S}$. Here and from now on, we write $S' = S \otimes_k k'$ and $\mathcal{V} = \mathcal{U}_{O_{k'}, S'} = \mathcal{U} \times_{O_{k, S}} O_{k', S'}$.

Given a finite non-trivial extension F/k , take $I \subset \Omega_k$ to be the set of places of k that split completely in the compositum $F \cdot k'$, according to Chebotarev's density theorem I is infinite.

Let \mathbf{E} be the finite set of conjugation classes of $H = \text{Gal}(\mathcal{U}'/\mathcal{V})$. Since U' is geometrically integral over k' , the geometric Chebotarev's density theorem (cf. [Eke90, Lem. 1.2]) allow us to construct an injection

$$\gamma : \mathbf{E} \longrightarrow ((\Omega_k \setminus S) \cap I) \otimes_k k'$$

such that for every $c \in \mathbf{E}$ there exists a point of finite residue field $\bar{x}_c \in \mathcal{V}(k'(\gamma(c)))$ with associated Frobenius element $\text{Frob}_{\bar{x}_c}$ belonging to the class c . Moreover, we can require that after restricting to k the images of different classes $c \in \mathbf{E}$ are different from each other.

For each $v \in S$, the effective 0-cycle z_v is defined by a separable polynomial $f_v \in k_v[t]$, in other words $\text{div}_{\mathbb{P}^1_v}(f_v) = z_v - d\infty$.

We denote v_c the place of k below $w' = \gamma(c)$, as v_c belongs to I we have $k_{v_c} = k'_{w'}$ and $k(v_c) = k'(w')$. By Hensel's lemma, the point \bar{x}_c lifts to a point $x_c \in \mathcal{V}(O_{w'}) \subset U_{k'}(k'_{w'}) = U(k_{v_c})$ where $O_{w'}$ is the ring of integers of the local field $k'_{w'}$. According to the lemma 3.6 below, there exists a closed point x'_c of U of degree $d-1$ different from x_c and from the P_i 's. We write $z_{v_c} = x_c + x'_c$ and $\text{div}_{\mathbb{P}^1_{v_c}}(f_{v_c}) = z_{v_c} - d\infty$ for a polynomial $f_{v_c} \in k_{v_c}[t]$.

Similarly, we take a place $v_0 \in I \setminus S$ away from the v_c 's and we write $\text{div}_{\mathbb{P}^1_{v_0}}(f_{v_0}) = z_{v_0} - d\infty$ with a closed point $z_{v_0} \in U$ different from the P_i 's and an irreducible polynomial $f_{v_0} \in k_{v_0}[t]$ of degree d .

We consider the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d\infty - \sum_{i=1}^n P_i) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d\infty) \longrightarrow \bigoplus_{i=1}^n P_i \rightarrow 0$$

where by abuse of notation P_i denotes the skyscraper sheaf on \mathbb{P}^1 supported at P_i . As $d > \sum_{i=1}^n \deg(P_i) - 2$, we have $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d\infty - \sum_{i=1}^n P_i)) = 0$ by Serre's duality and hence an exact sequence of global sections

$$0 \longrightarrow \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d\infty - \sum_{i=1}^n P_i)) \longrightarrow \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d\infty)) \xrightarrow{ev} \bigoplus_{i=1}^n k_i \rightarrow 0.$$

Here the k -linear map ev is the evaluation of a polynomial (of degree d) at points P_i . We fix a k -linear section $\sigma = \bigoplus_{i=1}^n \sigma_i$ of ev . By enlarging S if necessary, we may also assume that entries of the matrix of the linear map σ_i are all S -integers.

For $v \in S \cup \{v_c\}_{c \in \mathbf{E}} \cup \{v_0\}$, the polynomial f_v is written in a unique way as

$$f_v = f_{0v} + \sum_i \sigma_i(f_v(P_i))$$

with $f_{0v} \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d\infty - \sum_{i=1}^n P_i)) \otimes_k k_v$ a polynomial of degree d such that $f_{0v}(P_i) = 0$ for all i .

A priori $\rho_{i,v} = f_v(P_i)$ lies in $k_i \otimes_k k_v$, we have $\rho_{i,v} \in (k_i \otimes_k k_v)^*$ since $P_i \notin \text{supp}(z_v)$. Note that $I \setminus (S \cup \{v_c\}_{c \in \mathbf{E}} \cup \{v_0\})$ is infinite. Thanks to a generalized Dirichlet's theorem (cf. [San82, Cor. 4.4]), for each i there exists an element $\rho_i \in k_i^*$ and a place $w_i \in \Omega_{k_i}$ outside S such that

- ρ_i is sufficiently close to $\rho_{i,v}$ for all $v \in S \cup \{v_c\}_{c \in \mathbf{E}} \cup \{v_0\}$,
- $w_i(\rho_i) = 1$,

- ρ_i is a unit outside $\{w_i\} \cup (S \cup I) \otimes_k k_i$.

It suffices to find a closed point $\theta \in \mathbb{A}^1$ defined by $f \in k[t]$ having the desired properties (2)(3)(4) and such that $f(P_i) = \rho_i$ for all i .

Strong approximation property applied to the finite dimensional k -linear space $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d\infty - \sum_{i=1}^n P_i))$ give us a polynomial f_0 of degree d with coefficients in $\mathcal{O}_{S \cup I}$ such that $f_0(P_i) = 0$ for all i and such that f_0 is sufficiently close to f_{0v} for all $v \in S \cup \{v_c\}_{c \in \mathbf{E}} \cup \{v_0\}$. Then the degree d polynomial $f = f_0 + \sum_i \sigma_i(\rho_i) \in k[t]$ is sufficiently close to f_v with $S \cup I$ -integral coefficients. As f_{v_0} is irreducible, Krasner's lemma implies that f is irreducible over k_{v_0} and *a fortiori* irreducible over k . Moreover v_0 splits completely in k' , the field k' is contained in k_{v_0} , thus f is irreducible over k' . We write $\text{div}_{\mathbb{P}^1}(f) = \theta - d\infty$, then the 0-cycle θ is actually a closed point of U , and its preimage θ' in $U_{k'}$ is also a closed point. Let $L = k(\theta)$ be the residue field of θ , then $L' = L \otimes_k k'$ is a field and is the residue field of θ' . As a 0-cycle θ is sufficiently close to $z_{v_c} = x_c + x'_c$ for all $c \in \mathbf{E}$. This means that there exists a place w of $L = k(\theta)$ above v_c such that L_w/k_{v_c} is a trivial extension and moreover the image θ_w of θ under $U(L) \rightarrow U(L_w)$ is sufficiently close to x_c . Hence θ is also integral (for the model \mathcal{U}) at w , and its reduction modulo w is $\bar{x}_c \in \mathcal{U}(L(w)) = \mathcal{U}(k(v_c))$.

Recall that $w' = \gamma(c) \in \Omega_{k'}$ is a place over $v_c \in \Omega_k$ such that $k'_{w'}/k_{v_c}$ is a trivial extension. Let λ be a place of L' above $w \in \Omega_L$ and above $w' \in \Omega_{k'}$. The point $\theta' \in U_{k'}(L'_\lambda)$ is actually an integral point (for the model \mathcal{V}) of reduction modulo λ exactly $\bar{x}_c \in \mathcal{V}(L'(\lambda)) = \mathcal{V}(k'(w'))$. Consider the map $\text{Gal}(\bar{L}'/L') \rightarrow H = \text{Gal}(U'/U_{k'})$ defined by a chosen lifting of $\text{Spec}(\bar{L}') \rightarrow \text{Spec}(L') \xrightarrow{\theta'} U_{k'}$ to U' , the fact that the reduction of θ' is \bar{x}_c means that the conjugate class $c \in \mathbf{E}$ of the Frobenius element $\text{Frob}_{\bar{x}_c} \in H$ intersects with the image of $\text{Gal}(\bar{L}'/L')$. This holds for all $c \in \mathbf{E}$, then $\text{Gal}(\bar{L}'/L')$ maps surjectively to H by a well known result of finite group theory, cf. [Eke90, Lem. 1.1]. The preimage of θ' under $U' \rightarrow U_{k'}$ is then connected by the theory of étale fundamental groups. The point $\theta \in U$ has connected preimage in U' and so $\theta \in \text{Hil}' \subset \text{Hil}$. And it is an $S \cup I$ -integer since f has coefficients in $\mathcal{O}_{S \cup I}$. \square

Lemma 3.6. *Let k be a p -adic local field and D a finite set of closed points of \mathbb{P}^1/k . Then for every positive integer n , there exists a closed point of $\mathbb{P}^1 \setminus D$ of degree n .*

Proof. Over a local field, there exists infinite many irreducible polynomials of a fixed degree n . \square

3.5. Existence of local points.

Lemma 3.7 ([CTSSD98, Lem. 1.2]). *Let k be a number field and $\text{Spec}(O)$ a non-empty open set of the ring of integers of k . Let $\Pi : \mathcal{X} \rightarrow \mathbb{P}_O^1$ be a flat, projective morphism with \mathcal{X} regular and smooth over O . Let $\pi : X \rightarrow \mathbb{P}^1$ be the restriction of Π over $\text{Spec}(k)$. Let $T \subset \mathbb{P}_O^1$ be a closed subset, finite and étale over O , such that fibers of Π above points not in T are split. Let $T = \bigcup_{i=1}^n T_i$ be the decomposition of T into irreducible closed subsets, and let k_i be the field of fractions of T_i .*

After inverting finitely many primes in O , the following holds.

(a) *Given any closed point $u \in \mathbb{P}_O^1$, if the fiber \mathcal{X}_u over the finite field $k(u)$ is split, then it contains a smooth $k(u)$ -point.*

(b) *Given any closed point $\theta \in \mathbb{P}^1$, with Zariski closure $\text{Spec}(\tilde{O}) \simeq \tilde{\theta} \subset \mathbb{P}_O^1$, where \tilde{O}/O is finite with $\text{Frac}(\tilde{O}) = k(\theta)$, we denote by \tilde{O}' the integral closure of*

\tilde{O} in $k(\theta)$. If $u \in \tilde{\theta}$ is a closed point such that $\mathcal{X}_u/k(u)$ is split, then X_θ contains a smooth $k(\theta)_v$ -point where v is a place of $k(\theta)$ (associated to a closed point of $\text{Spec}(\tilde{O}')$) above u .

(c) Let u belong to one of the T_i 's, thus defining a place v_i of k_i . Assume that there exists a irreducible component Z of the fiber of Π at $P_i = T_i \times_O k$ which has multiplicity one. Let k'_i denote the algebraic closure of k_i in the function field of Z . If the place v_i splits completely in the ring of integers of k'_i , then $\mathcal{X}_u/k(u)$ is split.

(d) Assume that for each i there exists at least one irreducible component of $\Pi^{-1}(P_i)$ which has multiplicity one. Then given any finite extension F_0/k there exists a finite extension F containing F_0 such that for each place $v \in \Omega_k$ splits completely in F the induced map $X(L) \rightarrow \mathbb{P}^1(L)$ is surjective for all finite extensions L/k_v .

Proof. This is a version for closed points of [CTSSD98, Lem. 1.2], as indicated in [CTSSD98, page 20] the same proof works for this case. The last conclusion is slightly different from the original one, but the original proof applies.

We remark that by taking a certain non-empty open subset Y of X and extending it to an integral model over O , the same argument as in the proof (Lang-Weil's estimation and Hensel's lemma) implies that we can moreover require that the rational points over local fields that obtained are situated inside Y and are the liftings (by Hensel's lemma) of rational points of finite residue fields of the integral model. We will use this more precise statement in the proof of the main theorem. \square

4. PROOF OF THE MAIN THEOREM

The whole section is devoted to proving the main theorem 2.1.

Proof of Theorem 2.1. We are going to give a complete proof of the conclusion (2) with assumption (2) concerning 0-cycles of degree 1, similar argument works with assumption (2) concerning rational points with the help of [Wit12, Lem. 1.8]. The same method proves (1). Note that if $CH_0(X_v) \rightarrow CH_0(\mathbb{P}_v^1)$ is assumed injective for almost all places, the first conclusion in (3) is reduced to the conclusion (2), and the exactness of (E) is deduced by [Wit12, Prop. 3.1] applied with the base curve the projective line.

Outline of the proof.

After some preparations of Steps 1-3, in Step 4 we start from $\{z_v\} \perp Br(X)$. By Harari's formal lemma, we reduce the pairing to a finite sum over S_2 containing the places in S over which we need to approximate the local 0-cycles. We argue with moving lemmas to reduce to separable effective 0-cycles without changing the Brauer-Manin pairing. In Step 6 we apply Hilbert's irreducibility theorem (Prop. 3.5) to find a closed point $\theta \in \mathbb{P}^1$ close enough to the projections of the separable effective local 0-cycles. In Step 7, we verify that X_θ has points M_w locally everywhere. For the places in S_2 , we apply the implicit function theorem. For the places outside S_2 , We make use of the orthogonality given by the vertical elements $A_{i,j}$ in the Brauer group, these local points are essentially given by Hensel's lemma and Lang-Weil's estimation (Lem. 3.7). In Step 8, we compute each local term of the Brauer-Manin pairing and we modify certain M_w (by applying the geometric Chebotarev density theorem) in order to obtain, on the fiber X_θ , the orthogonality

to the Brauer group. The modified M_w 's are above carefully chosen places v_r in *Step 5*. The finite set E constructed in *Step 1* is used to verify one of the hypotheses of Chebotarev's theorem.

Aim.

In order to prove (2), we fix an integer δ , a positive integer N , and a finite set S of places of k . Given a family of local 0-cycle $\{z_v\}_{\Omega_k}$ of degree δ orthogonal to $Br(X)$, we need to find a global 0-cycle z such that z and z_v have the same image in $CH_0(X_v)/N$ for all $v \in S$.

Notation.

We preserve the notation Y, Z, X_r , and Λ , at the beginning of §2, the complement of $Z = \bigsqcup_{r=1}^l X_r$ is an open $Y \subset X$ with $\Lambda \subset Br(Y)$ generating $Br(X_\eta)/Br(k(t))$. Let P_i ($i = 1, \dots, n$) be the closed points of \mathbb{P}^1 such that the fiber $X_i = X_{P_i}$ is not split. Then for all i , the point P_i is contained in the open subset $V = \pi(Y)$ by the hypothesis (Br). Let U be a non-empty open subset of \mathbb{P}^1 such that all fibers of π over U is smooth and geometrically integral. Chose a k -point ∞ in $U \cap V$ different from those m_r 's. The open $V_0 = V \setminus \{\infty\}$ is contained in $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$. The fiber X_∞ is a smooth k -variety of function field $K_\infty = k(X_\infty)$. We write $Y_0 = Y \setminus X_\infty = \pi^{-1}(V_0)$.

For each $1 \leq r \leq l$, the fiber X_r at the closed point $m_r \in \mathbb{A}^1$ is split. We fix an irreducible component X_r^{irr} of X_r of multiplicity 1 such that it is geometrically integral over $k_r = k(m_r)$. The point m_r is defined by a monic irreducible polynomial $Q_r(t) \in k[t]$ and its residue field $k_r = k[t]/(Q_r(t))$. Denote by $K_r = k_r(X_r^{irr})$ the function field of X_r^{irr} . Let A be an element of $Br(k(X))$, we denote by $\partial_{A,r} \in H^1(K_r, \mathbb{Q}/\mathbb{Z})$ its residue at the generic point of X_r^{irr} . The subgroup of $H^1(K_r, \mathbb{Q}/\mathbb{Z})$ generated by the elements $\partial_{A,r}$ ($A \in \Lambda$) is of the form $G_r = H^1(Gal(K'_r/K_r), \mathbb{Q}/\mathbb{Z})$ where K'_r is a finite Abelian extension of K_r . Let k'_r be the algebraic closure of k_r in K'_r . Since k_r is algebraically closed in K_r , the subgroup $G'_r = H^1(Gal(K_r k'_r/K_r), \mathbb{Q}/\mathbb{Z})$ of G_r is isomorphic to $H^1(Gal(k'_r/k_r), \mathbb{Q}/\mathbb{Z})$. Let Y_r be a smooth open subset of X_r^{irr} with empty intersection with other irreducible components of X_r . By shrinking Y_r if necessary, we may assume that there exists a finite étale connected Galois cover W_r of Y_r of Galois group $Gal(K'_r/K_r)$ such that $W_r \rightarrow Y_r$ factorizes through $Y'_r = Y_r \times_{k_r} k'_r \rightarrow Y_r$. Then W_r is a variety geometrically integral over k'_r . Moreover, we may also assume that the elements of $H^1(Gal(K'_r/K_r), \mathbb{Q}/\mathbb{Z}) \subset H^1(K_r, \mathbb{Q}/\mathbb{Z})$ come from elements of $H_{\text{ét}}^1(Y_r, \mathbb{Q}/\mathbb{Z})$.

For each $1 \leq i \leq n$, let k_i be the residue field of P_i . The point P_i gives a rational point $e_i \in \mathbb{A}^1(k_i) = k_i$. We set $g'_i = t - e_i \in k_i[t]$ and $g_i = N_{k_i(t)/k(t)}(g'_i) \in k[t]$, then $k_i = k[t]/(g_i)$. We fix an irreducible component X_i^{irr} of the fiber X_i at P_i of multiplicity 1 which splits after an Abelian extension of k_i . Denote by $K_i = k_i(X_i^{irr})$ the function field of X_i^{irr} and by k'_i the algebraic closure of k_i in K_i . The field k'_i is a finite Abelian extension of k_i , it is a compositum of finitely many cyclic extension $k_{i,j}$ of k_i . By fixing a character χ of the cyclic group $Gal(k_{i,j}(t)/k_i(t))$, we define $(k_{i,j}/k_i, g'_i) = (\chi, g'_i)$ an element of $Br(k_i(t))$ via cup product, and we set $A_{i,j} = \text{cores}_{k_i(t)/k(t)}(k_{i,j}/k_i, g'_i) \in Br(k(t))$, cf. [CTSD94, §1]. There exists a closed subset $D \subset \mathbb{P}^1$ containing all P_i 's such that $A_{i,j} \in Br(\mathbb{P}^1 \setminus D)$ for all i, j .

Replacing by a smaller generalized Hilbertian subset if necessary, according to Proposition 3.4, we may assume that for every closed point $\theta \in \text{Hil} \subset \mathbb{A}^1$ the

specialization to the smooth fiber X_θ

$$sp_\theta : \frac{Br(X_\eta)}{Br(k(t))} \longrightarrow \frac{Br(X_\theta)}{Br(k(\theta))}$$

is an isomorphism, hence Λ maps onto the Brauer group of the fiber X_θ (up to constant elements which do not contribute to the Brauer-Manin obstruction on the fiber X_θ).

Step 1. Construction of a finite set E of elements in $Br(k(t))$.

The finite set E is constructed in order to verify (in *Step 8.4*) one of the hypotheses of the geometric Chebotarev's density theorem applied in *Step 8.5*.

Consider Faddeev's exact sequence (cf. [GS06, Cor. 6.4.6])

$$0 \longrightarrow Br(k) \longrightarrow Br(k(t)) \xrightarrow{\partial_\theta} \bigoplus_{\theta} H^1(k(\theta), \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

where the direct sum is taken over all closed points θ of \mathbb{A}^1 and ∂_θ is the residue map at θ . For each $1 \leq r \leq l$, we consider the subgroup

$$E'_r = \{\beta \in Br(k(t)); \partial_\theta(\beta) = 0 \text{ if } \theta \neq m_r \text{ and } \partial_{m_r}(\beta) \in G'_r = H^1(Gal(k'_r/k_r), \mathbb{Q}/\mathbb{Z})\}.$$

As $E'_r/Br(k)$ is a finite group, we take a finite subset $E_r \subset Br(k(t))$ of its representatives, then we define E to be the (disjoint) union of E_r 's. Thus $E \subset Br(V_0)$ and its image in $Br(k(X))$ is contained in $Br(Y_0)$. We often identify E with its image in $Br(k(X))$.

Consider the residue $\partial_{A,\infty}$ of $A \in \Lambda \cup E$ in $H^1(K_\infty, \mathbb{Q}/\mathbb{Z})$ at the generic point of X_∞ , we may choose a non-empty open subset Y_∞ of X_∞ such that these elements come from elements in $H^1_{\text{ét}}(Y_\infty, \mathbb{Q}/\mathbb{Z})$.

Step 2. Construction of a finite extension F_0/k .

By Tsen's theorem, the Brauer group of $\bar{k}(t)$ is trivial. Therefore there exists a finite extension k'_∞ of k such that the restriction of E in $Br(k'_\infty(t))$ is 0. We fix a finite extension F_0 of k containing k'_∞ , k'_i for all i , and k_r for all r . This is for the future computation of the Brauer-Manin pairing in *Step 8.3*.

Step 3. Extension to integral models.

There exists a finite set S_1 of places of k containing S such that all the following statements hold.

- S_1 contains all archimedean places of k .
- The variety X extends to an integral model \mathcal{X} over O_{k,S_1} , the morphism π extends to a flat morphism $\Pi : \mathcal{X} \rightarrow \mathbb{P}^1_{O_{k,S_1}}$.
- For each $1 \leq i \leq n$, the polynomial g_i has coefficients in O_{k,S_1} . The Zariski closure T_i of P_i in $\mathbb{P}^1_{O_{k,S_1}}$ (it is also the closure of the closed subvariety of $\mathbb{A}^1_{O_{k,S_1}}$ defined by $g_i = 0$) is étale over $\text{Spec}(O_{k,S_1})$. Non-split fibers of Π are situated above $T = \bigcup T_i$. Moreover T_i and $T_{i'}$ are disjoint if $i \neq i'$.
- For each $1 \leq r \leq l$, we denote by \tilde{m}_r the Zariski closure of m_r in $\mathbb{P}^1_{O_{k,S_1}}$ and by \mathcal{X}_r the fiber $\Pi^{-1}(\tilde{m}_r)$. The polynomial $Q_r(t)$ defining m_r has coefficients in O_{k,S_1} . The schemes \tilde{m}_r and $\tilde{m}_{r'}$ are disjoint if $r \neq r'$. The open subset $Y_r \subset X_r$ extends to an open $\mathcal{Y}_r \subset \mathcal{X}_r$ such that the elements of $H^1(Gal(K'_r/K_r), \mathbb{Q}/\mathbb{Z})$ come from elements of $H^1_{\text{ét}}(\mathcal{Y}_r, \mathbb{Q}/\mathbb{Z})$. The scheme \mathcal{Y}_r is smooth over O_{k,S_1} .

- We denote by $\widetilde{\infty}$ the Zariski closure of ∞ in $\mathbb{P}_{O_{k,S_1}}^1$ and by \mathcal{X}_∞ the fiber $\Pi^{-1}(\widetilde{\infty})$. The scheme $\widetilde{\infty}$ is disjoint with \widetilde{m}_r for all r . The open subset $Y_\infty \subset X_\infty$ extends to an open $\mathcal{Y}_\infty \subset \mathcal{X}_\infty$ such that the elements $\partial_{A,\infty}$ (where $A \in \Lambda \cup E$) come from elements of $H_{\text{ét}}^1(\mathcal{Y}_\infty, \mathbb{Q}/\mathbb{Z})$. The scheme \mathcal{X}_∞ is smooth over O_{k,S_1} .
- We set $\mathcal{V} = \mathbb{P}_{O_{k,S_1}}^1 \setminus \bigcup \widetilde{m}_r$, $\mathcal{V}_0 = \mathcal{V} \setminus \widetilde{\infty}$, $\mathcal{Y} = \mathcal{X} \setminus \bigcup \mathcal{X}_r$, and $\mathcal{Y}_0 = \mathcal{Y} \setminus \mathcal{X}_\infty$. The elements of $\Lambda \cup E \subset Br(Y_0)$ come from elements of $Br(\mathcal{Y}_0)$.
- The finite connected étale Galois cover $W_r \rightarrow Y_r$ extends to an O_{k_r,S_1} -morphism $\mathcal{W}_r \rightarrow \mathcal{Y}_r$ factorizing through $\mathcal{Y}'_r = \mathcal{Y}_r \times_{O_{k_r,S_1}} O_{k'_r,S_1} \rightarrow \mathcal{Y}_r$ such that $\mathcal{W}_r \rightarrow \mathcal{Y}_r$ is a Galois cover of group $Gal(K'_r/K_r)$ and $\mathcal{W}_r \rightarrow \mathcal{Y}'_r$ is a Galois cover of group $Gal(K'_r/K_r k'_r)$.
- Lemma 3.7 applies to $O = O_{k,S_1}$.
- For $1 \leq r \leq l$, the geometric Chebotarev's density theorem [Eke90, Lem. 1.2] applies to the $O_{k'_r,S_1}$ -morphism $\mathcal{W}_r \rightarrow \mathcal{Y}'_r$, i.e. the cardinality of $k'_r(v_r)$ for all $v_r \in \Omega_{k'_r} \setminus S_1 \otimes_k k'_r$ is large enough to guarantee the existence of a $k'_r(v_r)$ -point of \mathcal{Y}'_r having its Frobenius element in a given conjugacy class of $Gal(\mathcal{W}_r/\mathcal{Y}'_r) = Gal(K'_r/K_r k'_r)$.

Step 4. Application of formal lemma and moving lemmas.

Fix a closed point z_0 of X such that the 0-cycle $y_0 = \pi_*(z_0)$ is a closed point having the same residue field as z_0 and such that y_0 is different from ∞ and the points in D , denote by d_0 the degree of z_0 .

Let $\{z_v\}_{v \in \Omega_k}$ be a family of local 0-cycles of degree δ orthogonal to $Br(X)$. By Lemma 3.2, we may assume that each z_v is supported in Y_0 and $\pi_*(z_v)$ is supported disjoint from D . According to the formal lemma 3.1 applied to $\pi^*(A_{i,j}) \in Br(\pi^{-1}(\mathbb{P}^1 \setminus D))$ and the elements in $\Lambda \cup E \subseteq Br(Y_0)$, there exists a finite set of places S_2 of k containing S_1 such that and for each $v \in S_2$ a 0-cycle z'_v of degree δ supported in Y_0 and away from fibers above D such that

- $\sum_{v \in S_2} inv_v(\langle A_{i,j}, z'_v \rangle_v) = 0$ for all $A_{i,j}$;
- $\sum_{v \in S_2} inv_v(\langle A, z'_v \rangle_v) = 0$ for all $A \in \Lambda \cup E$;
- $z'_v = z_v$ for all $v \in S_1$.

In the rest of the proof, it suffice to approximate z'_v for every $v \in S_2$. We write in a unique way $z'_v = z_v^+ - z_v^-$, where z_v^+ and z_v^- are effective 0-cycles with support disjoint from each other. Recall that we are considering the images of 0-cycles in $CH_0(X_v)/N$, let a be a positive integer which is divisible by N and who annihilates all the elements $A \in \Lambda \cup E$ and $A_{i,j}$. Then the 0-cycle $z_v^1 = z'_v + ad_0 z_v^- = z_v^+ + (ad_0 - 1)z_v^-$ is effective of degree congruent to δ modulo ad_0 . We add to each z_v^1 a suitable positive multiple of az_0 and obtain z_v^2 of the same degree d for all $v \in S_2$. Moreover $d \equiv \delta \pmod{ad_0}$ can be taken to be sufficiently large such that Proposition 3.5 applies. By Lemma 3.3, for each $v \in S_2$, there exists an effective 0-cycle τ_v of degree d close enough to z_v^2 and such that it is supported in Y_0 and away from the fibers above D and such that $\pi_*(\tau_v)$ is a separable 0-cycle. By continuity of the Brauer-Manin pairing, note that a annihilates the elements appeared in the pairing, we have

- $\sum_{v \in S_2} inv_v(\langle A_{i,j}, \tau_v \rangle_v) = 0$ for all $A_{i,j}$;
- $\sum_{v \in S_2} inv_v(\langle A, \tau_v \rangle_v) = 0$ for all $A \in \Lambda \cup E$.

By [Wit12, Lem. 1.8], note that N divides a , the 0-cycles z'_v and τ_v have the same image in $CH_0(X_v)/N$ for all $v \in S_2$.

Step 5. Construction of assisting places v_r .

In this *Step* we construct for each $r \in \{1, \dots, l, \infty\}$ an assisting place v_r with a local 0-cycle, these will give us ability to modify the local points in *Step 8*.

Lemma 3.7(d) applied to our fibration with F_0 constructed in *Step 2* gives us a finite extension F/k containing F_0 . Suppose that the generalized Hilbertian subset Hil is given by a finite morphism $Z \rightarrow \mathbb{P}^1$, let k' be the algebraic closure of k inside the Galois closure of the field extension $k(Z)/k(t)$.

For each $r \in \{1, \dots, l, \infty\}$, let v_r be a place of k outside S_2 and which splits completely in $F \cdot k'$. We may also suppose that they are different from each others.

As consequence, for $1 \leq r \leq l$ the polynomial $Q_r(t)$ modulo v_r has a simple root in $k(v_r)$ which lifts to a k_{v_r} -point x_r of \mathbb{A}^1 satisfying $v_r(Q_r(x_r)) = 1$, moreover x_v can be chosen different from all the P_i 's. Similarly, we choose $x_\infty \in k_{v_\infty}^* \subset \mathbb{A}^1(k_{v_\infty})$ different from the P_i 's and such that $v_\infty(1/x_\infty) = 1$. Lemma 3.6 permit us to chose a closed point x'_r of \mathbb{A}^1 of degree $d-1$ different from the P_i 's, then $x_r + x'_r$ is a separable effective 0-cycle of degree d .

Step 6. Application of Hilbert's irreducibility theorem.

With the field F , the closed points P_i , and the generalized Hilbertian subset $\text{Hil} \cap V_0$, we apply Proposition 3.5 to local 0-cycles $\pi_*(\tau_v)$ for $v \in S_2$ as well as $x_r + x'_r$ for $r \in \{1, \dots, l, \infty\}$. By Chebotarev's density theorem, we obtain an infinite set I of the places of k which split completely in F , a *fortiori* split completely in k_r , $k_{i,j}$ and k'_∞ . According to the construction of I in the proof of 3.5, the places v_r belong to I for $r \in \{1, \dots, l, \infty\}$. We also obtain a closed point $\theta \in \text{Hil} \subset V_0 \subset \mathbb{A}^1$ of degree d sufficiently close to $\pi_*(\tau_v)$ for $v \in S_2$ and sufficiently close to $x_r + x'_r$ for $r \in \{1, \dots, l, \infty\}$, as a $k(\theta)$ -point of \mathbb{A}^1 it is an $S_2 \cup I$ -integer. Moreover, let $f \in k[t]$ be the polynomial defining θ constructed in Proposition 3.5, for each $1 \leq i \leq n$ we obtain a place w_i of k_i away from $S_2 \cup I$ such that $w_i(f(P_i)) = 1$ and $w(f(P_i)) = 0$ for all $w \in \Omega_{k_i} \setminus (S_2 \cup I) \otimes_k k_i$ different from w_i .

Step 7. The fiber X_θ has points locally everywhere.

Let $w \in \Omega_{k(\theta)}$ be a place above $v \in \Omega_k$.

If $v \in S_2$, the implicit function theorem shows that X_θ possesses $k(\theta)_w$ -points.

If $v \in I$, Lemma 3.7(d) implies that X_θ possesses $k(\theta)_w$ -points.

If $v \notin S_2 \cup I$, we denote by $\tilde{\theta} \simeq \text{Spec}(A)$ the Zariski closure of θ in $\mathbb{P}_{O_{k,S_2}}^1$, where A is a finite O_{k,S_2} -algebra with fraction field $k(\theta)$ and its integral closure in $k(\theta)$ is $O_{k(\theta),S_2}$. We fix a place w of $k(\theta)$ above v , it defines closed point w of the normalization $\text{Spec}(O_{k(\theta),S_2})$ of $\tilde{\theta}$ lying above a certain closed point $w_\theta \in \tilde{\theta}$. Recall that $\tilde{\theta}$ and T_i are locally defined respectively by f and g_i (polynomials with $O_{S_2 \cup I}$ -integral coefficients). There are two possible cases.

- (i) If w_θ is contained in one (unique) of the T_i 's. We know that for $w' \in \Omega_{k_i} \setminus (S_2 \cup I) \otimes_k k_i$, we have $w'(f(P_i)) = 0$ except only one possible case where $w' = w_i$ and $w_i \in \Omega_{k_i} \setminus (S_2 \cup I) \otimes_k k_i$, for which we have $w_i(f(P_i)) = 1$. The point w_θ is contained in T_i if and only if the exceptional case happens. In such a case, considering the intersection $T_i \cap \tilde{\theta}$ at the point w_i , the

intersection multiplicity equals to 1 since $w_i(f(P_i)) = 1$. Then w_i , viewed as a closed point w_θ of $\tilde{\theta}$, must be a regular point of $\tilde{\theta}$. Therefore $w = w_\theta = w_i$, $k_{iw_i} = k(\theta)_w$ and $w(g_i(\theta)) = w_i(f(P_i)) = 1$.

- (ii) If $w_\theta \notin T_i$ for all i , then the fiber $\mathcal{X}_{w_\theta}/k(w_\theta)$ is split by the construction of T_i , thus $X_\theta(k(\theta)_w) \neq \emptyset$ according to Lemma 3.7(b). In this case, we know that $g_i(\theta)$ is a unit (modulo w_θ) in $k(w_\theta) \subset k(w)$ since $w_\theta \notin T_i \cap \tilde{\theta}$, then $w(g_i(\theta)) = 0$.

Remark that if $w_i \in I \otimes_k k_i$, case (i) will never happen. To complete this *Step*, it remains to verify that for $w = w_i \in T_i$ (case (i) if it happens) the fiber X_θ possesses $k(\theta)_{w_i}$ -points. This will occupy the rest of this *Step* and we may assume that $w_i \in \Omega_{k_i} \setminus (S_2 \cup I) \otimes_k k_i$.

We define $E_i = k_i \otimes_k k(\theta)$ and $F_{i,j} = k_{i,j} \otimes_k k(\theta)$. Then

$$\langle A_{i,j}, \theta \rangle_{\mathbb{P}^1} = \text{cores}_{k(\theta)/k} \text{cores}_{E_i/k(\theta)}(F_{i,j}/E_i, g'_i(\theta)) \in Br(k)$$

by definition.

By continuity of the Brauer-Manin pairing,

$$\sum_{v \in S_2} \text{inv}_v(\langle A_{i,j}, \theta \rangle_v) = \sum_{v \in S_2} \text{inv}_v(\langle A_{i,j}, \pi_*(\tau_v) \rangle_v) = 0,$$

hence

$$\sum_{v \in \Omega_k \setminus S_2} \text{inv}_v(\langle A_{i,j}, \theta \rangle_v) = 0$$

since θ is global. In other words

$$\begin{aligned} \sum_{v \in \Omega_k \setminus S_2} \text{inv}_v(\text{cores}_{k(\theta)/k} \text{cores}_{E_i/k(\theta)}(F_{i,j}/E_i, g'_i(\theta))) &= 0, \\ \sum_{w \in \Omega_{k(\theta)} \setminus S_2 \otimes_k k(\theta)} \text{inv}_w(\text{cores}_{E_i/k(\theta)}(F_{i,j}/E_i, g'_i(\theta))) &= 0. \end{aligned}$$

We consider a place w of $k(\theta)$ away from S_2 , and we are going to calculate each term in the sum above.

If $w \in I \otimes_k k(\theta)$, let v be the place of k below w . By construction, the extension of local fields associated to $k_{i,j}/k_i$ is trivial above the place v , then the extension $F_{i,j}/E_i$ is trivial above the place w , we find that

$$\text{inv}_w(\text{cores}_{E_i/k(\theta)}(F_{i,j}/E_i, g'_i(\theta))) = 0.$$

If $w \notin I \otimes_k k(\theta)$ and $w \neq w_i$ (i.e., the point $w_\theta \in \tilde{\theta}$ associated to w is not in T_i), we recall that in this case $w(g_i(\theta)) = 0$, then $g_i(\theta) = N_{E_i/k(\theta)}(g'_i(\theta))$ is a unit at w , we also obtain

$$\text{inv}_w(\text{cores}_{E_i/k(\theta)}(F_{i,j}/E_i, g'_i(\theta))) = 0.$$

Therefore we get finally

$$(\star) \quad \text{inv}_{w_i}(\text{cores}_{E_i/k(\theta)}(F_{i,j}/E_i, g'_i(\theta))) = 0.$$

Consider the natural map $E_i \rightarrow E_i \otimes_{k(\theta)} k(\theta)_{w_i}$, where $E_i \otimes_{k(\theta)} k(\theta)_{w_i}$ is a product of extensions of $k(\theta)_{w_i}$. Note that $w(N_{E_i/k(\theta)}(g'_i(\theta))) = w(g_i(\theta))$ equals to either 0 or 1 according to $w \neq w_i$ or $w = w_i$, there is only one of these extensions, denoted by E_{i,w_i} , in which the image of $g'_i(\theta)$ is not a unit but a uniformizer, and moreover, $E_{i,w_i}/k(\theta)_{w_i}$ is trivial. The equality (\star) implies that $(F_{i,j}/E_i, g'_i(\theta)) \otimes_{E_i} E_{i,w_i} = 0$, we have then for all j the cyclic extension $k_{i,j}/k_i$ is trivial after $\otimes_{E_i} E_{i,w_i}$ since $g'_i(\theta)$ is a uniformizer of E_{i,w_i} , we find that k'_i/k_i is trivial after $\otimes_{E_i} E_{i,w_i}$. By Lemma

3.7(c), the reduction $\mathcal{X}_{w_i}/k(w_i)$ of X_θ modulo w_i is split, and X_θ contains a $k(\theta)_{w_i}$ -point by Lemma 3.7(b).

The following final remark will be used in the next *Step*. For all places $w \in \Omega_{k(\theta)}$ outside S_2 , the existence of $k(\theta)_w$ -points of X_θ is deduced as above by applying Lemma 3.7. In other words, these local points are in fact integral (with respect to w) points obtained by lifting of points of finite residue fields. As remarked in the proof of Lemma 3.7, they can be chosen to be integral points of the integral model $\mathcal{X} \setminus (\bigsqcup_r \mathcal{X}_r \setminus \mathcal{Y}_r)$ of the Zariski open $X \setminus (\bigsqcup_r X_r \setminus Y_r) \subset X$ where r runs through $\{1, \dots, l, \infty\}$.

Step 8. Orthogonality to the Brauer group.

For each $w \in \Omega_{k(\theta)}$, let M_w denote the $k(\theta)_w$ -point we found on the fiber X_θ in *Step 7*. We know by continuity of the Brauer-Manin pairing and the projection formula that

$$\sum_{w \in S_2 \otimes_k k(\theta)} \text{inv}_w(A(M_w)) = \sum_{v \in S_2} \text{inv}_v(\langle A, \tau_v \rangle_v) = 0$$

for all $A \in \Lambda \cup E$. In this *Step*, we are going to compute $\text{inv}_w(A(M_w))$ for w outside S_2 and modify certain M_w such that they satisfy the equality

$$\sum_{w \in \Omega_{k(\theta)}} \text{inv}_w(A(M_w)) = 0$$

for all $A \in \Lambda$.

Step 8.1. Classification of places of $k(\theta)$.

As a 0-cycle θ is sufficiently close to $x_r + x'_r$ for $1 \leq r \leq l$, this means that there exists a place w_r^0 of $k(\theta)$ above v_r such that the extensions $k(\theta)_{w_r^0}/k_{v_r}$ and $k(\theta)(w_r^0)/k(v_r)$ are trivial and the image of θ in $\mathbb{A}^1(k(\theta)_{w_r^0})$ is sufficiently close to x_r . Hence $w_r^0(Q_r(\theta)) = v_r(Q_r(\theta)) = v_r(Q_r(x_r)) = 1$, *a fortiori* $w_r^0(\theta) \geq 0$ since the coefficients of Q_r are integers at v_r . Similarly, there exists a place w_∞^0 of $k(\theta)$ above v_∞ such that $w_\infty^0(1/\theta) = 1$.

We consider the reduction of $\theta \in V_0 \subset \mathbb{P}^1$ modulo a place $w \in \Omega_{k(\theta)} \setminus S_2 \otimes_k k(\theta)$, there are three possibilities:

- (a) it is in \mathcal{V}_0 , if and only if $w(Q_r(\theta)) = 0$ (*a fortiori* $w(\theta) \geq 0$), we denote by $\Omega_0 \subset \Omega_{k(\theta)}$ the subset of such places;
- (b) it is in one (unique) of the \tilde{m}_r 's ($1 \leq r \leq l$), if and only if $w(Q_r(\theta)) > 0$ (*a fortiori* $w(\theta) \geq 0$), we denote by $\Omega_r \subset \Omega_{k(\theta)}$ the subset of such places, in particular $w_r^0 \in \Omega_r$;
- (c) it is in ∞ if and only if $w(\theta) < 0$, we denote by $\Omega_\infty \subset \Omega_{k(\theta)}$ the subset of such places. Then $\Omega_\infty \subset I \otimes_k k(\theta)$ since θ is an $S_2 \cup I$ -integer.

The subsets $\Omega_0, \Omega_1, \dots, \Omega_l, \Omega_\infty$ form a partition of $\Omega_{k(\theta)} \setminus S_2 \otimes_k k(\theta)$, and Ω_r is finite for $r \in \{1, \dots, l, \infty\}$.

Step 8.2. Computation of $A(M_w)$.

For $w \in \Omega_{k(\theta)} \setminus S_2 \otimes_k k(\theta)$, we want to compute $A(M_w)$ for $A \in \Lambda \cup E \subset Br(Y_0)$. By construction in *Step 7*, the point M_w is a lifting (by Hensel's lemma) of its modulo- w -reduction $M(w)$ of \mathcal{Y}_r according to $w \in \Omega_r$ ($r \in \{0, 1, \dots, l, \infty\}$).

We find, by [Har94, Cor. 2.4.3], for $A \in \Lambda \cup E$

- (a) $\text{inv}_w(A(M_w)) = 0$ if $w \in \Omega_0$;

- (b) $\text{inv}_w(A(M_w)) = w(Q_r(\theta)) \cdot \partial_{A,r,M(w)}(F_{r,M(w)})$ if $w \in \Omega_r (1 \leq r \leq l)$, where $\partial_{A,r,M(w)}$ is the evaluation of the element $\partial_{A,r} \in H_{\text{ét}}^1(\mathcal{Y}_r, \mathbb{Q}/\mathbb{Z})$ at the point $M(w)$ of \mathcal{Y}_r , and where $F_{r,M(w)} \in \text{Gal}(\mathcal{Y}'_r/\mathcal{Y}_r)$ is the Frobenius element at $M(w)$.
- (c) $\text{inv}_w(A(M_w)) = w(1/\theta) \cdot \partial_{A,\infty,M(w)}$ if $w \in \Omega_\infty$, where $\partial_{A,\infty,M(w)}$ is the evaluation of $\partial_{A,\infty} \in H_{\text{ét}}^1(\mathcal{Y}_\infty, \mathbb{Q}/\mathbb{Z})$ at $M(w) \in \mathcal{Y}_\infty$.

Step 8.3. Computation for $w \in \Omega_\infty$.

If $A \in \Lambda \subset \text{Br}(Y)$, as the point $\infty \in \pi(Y)$, the residue $\partial_{A,\infty}$ equals to 0 in $H^1(K_\infty, \mathbb{Q}/\mathbb{Z})$. If $A \in E$, it comes from an element of $\text{Br}(k(t))$ which becomes 0 in $\text{Br}(k'_\infty(t))$. Moreover, the place v of k below the place $w \in \Omega_\infty$ belongs to I , thus the field $k(\theta)_w$ contains $k_v = k'_{\infty,v'}$ for all v' above v , then $A = 0$ in $\text{Br}(k(\theta)_w(t))$. In both cases, $\text{inv}_w(A(M_w)) = 0$ for $w \in \Omega_\infty$.

Step 8.4. Computation for $w \in \Omega_r (1 \leq r \leq l)$.

A priori, the element $\sum_{w \in \Omega_r} w(Q_r(\theta)) \cdot F_{r,M(w)}$ (additive notation) lies in the Abelian group $\text{Gal}(K'_r/K_r)$, we are going to prove that this element belongs to the subgroup $\text{Gal}(K'_r/K_r k'_r)$.

Let ρ_r be an arbitrary element of $G'_r = H^1(\text{Gal}(K_r k'_r/K_r), \mathbb{Q}/\mathbb{Z}) \subset G_r$, then there exists an element A_r of $E \subset \text{Br}(k(t))$ whose only possible non-zero residue is at m_r giving value ρ_r . In this case, we have $\text{inv}_w(A_r(M_w)) = 0$ for all $w \in \Omega_{r'}$ if $r' \neq r$ and $1 \leq r' \leq l$. On the other hand, as the specialization of $A_r \in \text{Br}(k(X))$ at X_θ comes from $\text{Br}(k(\theta))$, we find the equality $\sum_{w \in \Omega_k(\theta)} \text{inv}_w(A_r(M_w)) = 0$. Combining the computations for places in Ω_0 and Ω_∞ done in *Steps 8.2 and 8.3*

$$\sum_{w \in \Omega_r} \text{inv}_w(A_r(M_w)) = 0,$$

in other words,

$$\sum_{w \in \Omega_r} \rho_r (w(Q_r(\theta)) \cdot F_{r,M(w)}) = 0$$

for all $\rho_r \in G'_r$. Hence the element $\sum_{w \in \Omega_r} w(Q_r(\theta)) \cdot F_{r,M(w)}$ lies in $\text{Gal}(K'_r/K_r k'_r)$.

Step 8.5. Modification of $M_{w_r^0} (1 \leq r \leq l)$.

Recall that there exists a place $w_r^0 \in \Omega_r$ above v_r with $w_r^0(Q_r(\theta)) = 1$, moreover the extensions $k(\theta)_{w_r^0}/k_{v_r}$ and $k(\theta)(w_r^0)/k(v_r)$ are trivial. As v_r splits completely in k'_r , the Frobenius element $F_{r,M(w_r^0)}$ lies in $\text{Gal}(K'_r/K_r k'_r)$.

The geometric Chebotarev's density theorem [Eke90, Lem. 1.2] implies the existence of a point $M'(w_r^0)$ of \mathcal{Y}_r whose Frobenius element $F_{r,M'(w_r^0)}$ is exactly the element $F_{r,M(w_r^0)} - \sum_{w \in \Omega_r} w(Q_r(\theta)) \cdot F_{r,M(w)} \in \text{Gal}(K'_r/K_r k'_r)$. We lift it to a $k(\theta)_{w_r^0}$ -point $M'_{w_r^0}$ of X_θ .

For each $1 \leq r \leq l$, we replace $M_{w_r^0}$ by $M'_{w_r^0}$ and we keep M_w for all $w \in \Omega_r \setminus \{w_r^0\}$. We verify that

$$\sum_{w \in \Omega_r} \text{inv}_w(A(M_w)) = 0$$

for $A \in \Lambda$ and for all $1 \leq r \leq l$. Therefore

$$\sum_{w \in \Omega_k(\theta)} \text{inv}_w(A(M_w)) = 0$$

for all $A \in \Lambda$.

Step 9. End of the proof.

The specialization map sp_θ maps Λ on to $Br(X_\theta)/Br(k(\theta))$, the family $\{M_w\}_{w \in \Omega_{k(\theta)}}$ of local points on X_θ is then orthogonal to $Br(X_\theta)$. The hypothesis gives a global 0-cycle z' of degree 1 on X_θ such that z' and M_w have the same image in $CH_0(X_{\theta,w})/N$ for all $w \in S_2 \otimes_k k(\theta)$. The 0-cycle z' is regarded as a 0-cycle of degree $d \equiv \delta \pmod{ad_0}$ on X , by subtracting a suitable multiple of az_0 from z' , we get a global 0-cycle z of degree δ on X such that z and z_v have the same image in $CH_0(X_v)/N$ for all $v \in S$.

□

Remark 4.1. In *Step 8* we considered elements representing $Br(X_\eta)/Br(k(t))$, and in *Step 7* we considered elements $A_{i,j}$ coming from $Br(k(t))$ as well. We have made use of almost the whole Brauer group.

5. CONCERN ABOUT THE HYPOTHESIS (Br)

If one could remove the hypothesis (Br) in the main theorem (which seems unlikely), as explained in the remark 2.2, Theorem 2.1 would apply to all proper smooth models of varieties defined by Abelian normic equations

$$N_{K/k}(\vec{x}) = P(t).$$

In two extreme cases, the hypothesis (Br) is automatically satisfied, cf. §5.1.

Compare to the recent result of Dasheng Wei [Wei], in one of his theorems concerning Abelian normic equations, he has assumed a hypothesis of the same type as (Br) and he has found explicit examples, cf. §5.2.

5.1. Previous results for 0-cycles.

5.1.1. If all the closed fibers of $\pi : X \rightarrow \mathbb{P}^1$ are split (in particular, if they are geometrically integral), the hypotheses (Ab-sp) and (Br) are automatically verified. This special case has been proved by the author in [Liac]. In [Liac] we also proved some theorems on fibrations over higher dimensional projective spaces, unfortunately the method does not extend directly to this work even if one can remove the assumption (Br). Elements of $Br(X) \subset Br(k(X))$ which are not coming from $Br(k(t))$ lead to difficulties in comparing Brauer groups appeared in the induction process.

5.1.2. The hypothesis (Br) is also automatically verified if the morphism $Br(k(t)) \rightarrow Br(X_\eta)$ is surjective (e.g. certain fibrations in Châtelet surfaces). By Proposition 3.4 the Brauer group $Br(X_\theta)$ consists only elements coming from $Br(k(\theta))$ for all $\theta \in \text{Hil}$. The condition (1) (resp. (2)) of the main theorem 2.1 becomes

- The $k(\theta)$ -variety X_θ satisfies the Hasse principle (resp. weak approximation) for rational points or for 0-cycles of degree 1.

This special case has also been proved by the author in [Liaa].

5.2. The recent result of Wei on normic equations. In a recent paper of Wei [Wei], by explicit computations of elements in the Brauer group he has proved that the Brauer-Manin obstruction is the only obstruction to the Hasse principle for 0-cycles of degree 1 on varieties defined by several specific normic equations. Two of them are closely related to the main result of this work, we discuss them in the following two subsections.

In his paper, the exactness of (E) (which involves 0-cycles of all degree) was not discussed and he dealt with only 0-cycles of degree 1. He needed a degree argument to assure that $k(\theta)$ and K are linearly disjoint over k . We remark that, by combining the argument of generalized Hilbertian subsets (in Proposition 3.5 no restriction of the degree is made), the sequence (E) can be proved exact at least for the following two cases.

5.2.1. Consider the equation over a number field k

$$N_{K/k}(\vec{x}) = P(t)$$

where K/k is a finite Abelian extension of degree n and $P(t) \in k[t]$ is a non-zero polynomial. Let X^{CTHS} be the CTHS partial compactification (named after the authors) of the closed sub-variety of \mathbb{A}^{n+1} defined by the equation, it admits a morphism $X^{CTHS} \rightarrow \mathbb{P}^1$, cf. [CTHS03, §2] for the construction. Let X be a smooth compactification of X^{CTHS} , the extended fibration $X \rightarrow \mathbb{P}^1$ satisfies (gen) and (Ab-sp).

We denote by T the norm one k -torus $R_{K/k}^1 \mathbb{G}_m$ defined by $N_{K/k}(\vec{x}) = 1$. Assuming an additional hypothesis that $\text{III}_\omega^2(\widehat{T})_P = \text{III}_\omega^2(\widehat{T})$, cf. [CTHS03, §2] for definition of these groups, Wei proved that the Brauer-Manin obstruction is the only obstruction to the Hasse principle for 0-cycles of degree 1 on X , [Wei, Thm. 4.1]. He gave a explicit example satisfying this additional hypothesis, [Wei, Cor. 3.4]. His example is essentially new in the sense that $\text{Gal}(K/k) \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is not cyclic. It is remarked that this additional hypothesis is equivalent to the surjectivity of $\text{Br}(X^{CTHS}) \rightarrow \text{Br}(X_\eta)/\text{Br}(k(t))$, cf. the remark before Corollary 3.4 in [Wei]. The latter is of the same type as the hypothesis (Br).

Combining our main theorem, we can prove the following slightly generalized statement for Abelian normic equations (0-cycles of all degrees and weak approximation are considered). Wei's example is certainly included.

Corollary 5.1. *Let $X \rightarrow \mathbb{P}^1$ be a smooth proper model of the Abelian normic equation $N_{K/k}(\vec{x}) = P(t)$ containing the CTHS partial compactification X^{CTHS} as an open subset. If the following hypothesis is verified*

(Br') *there exists an open subscheme $Y \subset X$ with complement $X \setminus Y$ a disjoint union of closed split fibers such that the subgroup $\text{Br}(Y) + \text{Br}(X^{CTHS})$ of $\text{Br}(X_\eta)$ maps onto $\text{Br}(X_\eta)/\text{Br}(k(t))$,*

then (E) is exact for X .

Proof. As in the proof of the main theorem 2.1, it suffices to verified the equality

$$\sum_{w \in \Omega_{k(\theta)}} \text{inv}_w(A(M_w)) = 0$$

where $A \in \Lambda \subset \text{Br}(Y) + \text{Br}(X^{CTHS})$ is of the form $A = A_1 + A_2$ with $A_1 \in \text{Br}(Y)$ and $A_2 \in \text{Br}(X^{CTHS})$. The group $\text{Br}(X^{CTHS})$ maps onto $\text{III}_\omega^2(\widehat{T})_P$ ([CTHS03, Prop. 2.5]), and elements of the latter have an explicit description [Wei, Lem. 3.2]. With the help of this description, the element A_2 gives no contribution to the Brauer-Manin pairing, i.e. $\sum_{w \in \Omega_{k(\theta)}} \text{inv}_w(A_2(M_w)) = 0$ for local points $M_w \in X_\theta(k(\theta)_w)$ which can be chosen arbitrarily for $w \notin S_2 \otimes_k k(\theta)$, cf. Proof of Theorem 4.1 in [Wei]. Then we modify certain M_w 's (with $w \notin S_2 \otimes_k k(\theta)$) such that $\sum_{w \in \Omega_{k(\theta)}} \text{inv}_w(A_1(M_w)) = 0$, cf. Step 8.5. This completes the proof. \square

5.2.2. In [CT], Colliot-Thélène computed the Brauer group of any smooth proper model of the equation

$$(x_1^2 - ax_2^2)(y_1^2 - by_2^2)(z_1^2 - abz_2^2) = c$$

with $a, b, c \in F^*$, over any field F of characteristic zero.

This permits us to consider the equation

$$(x_1^2 - ax_2^2)(y_1^2 - by_2^2)(z_1^2 - abz_2^2) = P(t)$$

over a number field k with $a, b \in k^*$ and $P(t) \in k[t]$ a non-zero polynomial. It has an (inexplicit) smooth proper model $X \rightarrow \mathbb{P}^1$ (parameterized by t). Its generic fiber X_η is in fact a compactification of a homogeneous space under an algebraic torus split by the Abelian extension $k(\sqrt{a}, \sqrt{b})/k$. The fibration satisfies (gen) and (Ab-sp). In this special case, [CT, Thm. 4.1] implies that for all but finitely many closed point $\theta \in \mathbb{P}^1$, the specialization $sp_\theta : Br(X_\eta) \rightarrow Br(X_\theta)/Br(k(\theta))$ maps the subgroup generated by the quaternion algebra $\mathcal{Q} = (x_1^2 - ax_2^2, b) \in Br(X_\eta)$ onto the group $Br(X_\theta)/Br(k(\theta)) \subseteq \mathbb{Z}/2\mathbb{Z}$. But we do not know the ramifications (with respect to residue maps) of the element \mathcal{Q} since we are not able to describe explicitly the model X . Fortunately, for all $w \in \Omega_{k(\theta)}$ outside a certain finite set S of places and for all $M_w \in X_\theta(k(\theta)_w)$ we have $inv_w(\mathcal{Q}(M_w)) = 0$, cf. the proof of Theorem 4.2 of [Wei]. Therefore in such a particular case, we can still prove the conclusions in the main theorem for X without verification of (Br), i.e. the sequence (E) is exact for smooth proper varieties defined by the equation above.

5.3. An open question. The assumption (Br) is indispensable in our approach, we ask the following question.

Open question. *For which finite Abelian extension K/k and for which polynomial $P(t) \in k[t]$, there exists a smooth proper model of the normic equation*

$$N_{K/k}(\vec{x}) = P(t)$$

satisfying the assumption (Br) or (Br')?

Answers to the question will give concrete examples of the main theorem.

6. A SIMILAR RESULT FOR RATIONAL POINTS

The work of Colliot-Thélène, Skorobogatov, and Sansuc [CTSSD98], on which the present work bases, shows the similarity between the behaviors of 0-cycles of degree 1 and of rational points on fibered varieties.

One may restate Theorem 2.1 simply by replacing the phrase “0-cycles of degree 1” by “rational points” admitting Schinzel’s hypothesis. However, replacing Salberger’s device by Schinzel’s hypothesis, the proof can not be transferred correctly. In fact, we need to compare the Brauer group of closed fibers X_θ with that of the generic fiber X_η . In order to deal with this, a certain generalized Hilbertian subset Hil must be involved. Proposition 3.5 shows that Salberger’s device does be fixed into the context of Hil. Instead, for the case of rational points, Schinzel’s hypothesis can not be easily combined with the Hilbertian subset condition, we may need a stronger hypothesis. While in [CTSSD98], only the vertical Brauer group of the fibration is considered, the similar problem does not appear, Salberger’s device and Schinzel’s hypothesis play the same role in the proof for 0-cycles and for rational points respectively.

Actually, this is the only difficulty. We restrict ourselves to Abelian normic equations, a special case of Theorem 2.1, this difficulty will disappear automatically as follows. Let X be a smooth proper model of the equation $N_{K/k}(\vec{x}) = P(t)$ containing the CTHS partial compactification X^{CTHS} , it is fibered as $X \rightarrow \mathbb{P}^1$ by the parameter t . Taking into account the fact that $H^3(k(t), \mathbb{G}_m) = 0$ (resp. $H^3(k(\theta), \mathbb{G}_m) = 0$), the Hochschild-Serre spectral sequence implies that $Br(X_\eta)/Br(k(t)) \simeq H^1(k(t), Pic(X_{\bar{\eta}}))$ (resp. $Br(X_\theta)/Br(k(\theta)) \simeq H^1(k(\theta), Pic(X_{\bar{\theta}}))$). In general, the specialization (cf. Prop. 3.4)

$$sp_\theta : H^1(k(t), Pic(X_{\bar{\eta}})) \rightarrow H^1(k(\theta), Pic(X_{\bar{\theta}}))$$

is an isomorphism for those closed points θ belonging to a certain mysterious generalized Hilbertian subset $Hil \subset \mathbb{P}^1$. In the case of Abelian normic equations, we can simply take Hil to be a dense open subset, on which the argument in [CTSSD98] with Schinzel's hypothesis works. In the following paragraph, we explain why Hil can be taken to be an open subset of \mathbb{P}^1 .

We consider the k -torus T defined by $N_{K/k}(\vec{x}) = 1$. Let T^c be the equivariant smooth compactification of T used in the construction of X^{CTHS} . By construction, the generic fiber X_η is the contracted product $E_\eta \times^{T_{k(t)}} T_{k(t)}^c$ where E_η is the torsor under $T_{k(t)}$ defined by $N_{K/k}(\vec{x}) = P(t)$. Respectively, for θ belonging to a certain open subset $U_0 \subset \mathbb{P}^1$ the closed fiber X_θ is the contracted product $E_\theta \times^{T_{k(\theta)}} T_{k(\theta)}^c$ where E_θ is the torsor under $T_{k(\theta)}$ defined by $N_{K/k}(\vec{x}) = P(\theta)$. According to [CTHS03, Lem. 2.1], we have isomorphic Galois modules $Pic(X_{\bar{\eta}}) \simeq Pic(T_{k(t)}^c)$ and $Pic(X_{\bar{\theta}}) \simeq Pic(T_{k(\theta)}^c)$ for $\theta \in U_0$. Note that for the question of rational points, we only need to consider fibers over rational points θ , i.e. $k(\theta) = k$. The specialization then becomes a homomorphism

$$H^1(k(t), Pic(T_{k(t)}^c)) \rightarrow H^1(k, Pic(\overline{T^c}))$$

induced by an equivariant homomorphism between Galois modules $Pic(T_{k(t)}^c) \rightarrow Pic(\overline{T^c})$, which is an isomorphism of Abelian groups. To see that the homomorphism between cohomologies is an isomorphism, let l/k be a finite Galois extension which splits T . Note that the Picard groups are torsion-free, the Hochschild-Serre spectral sequence shows that $H^1(k(t), Pic(T_{k(t)}^c)) \simeq H^1(l(t)/k(t), Pic(T_{l(t)}^c))$ and $H^1(k, Pic(\overline{T^c})) \simeq H^1(l/k, Pic(T_l^c))$. By identifying $Gal(l(t)/k(t))$ and $Gal(l/k)$, we see that

$$H^1(l(t)/k(t), Pic(T_{l(t)}^c)) \rightarrow H^1(l/k, Pic(T_l^c))$$

is an isomorphism. Hence the specialization sp_θ is an isomorphism for all $\theta \in U_0(k)$. The argument with Schinzel's hypothesis works and gives us the following theorem for rational points.

Theorem 6.1. *Let X be a smooth proper model of the Abelian normic equation $N_{K/k}(\vec{x}) = P(t)$, fibered as $X \rightarrow \mathbb{P}^1$ by the parameter t . Assume that Schinzel's hypothesis holds and moreover*

- (Br) *there exists a finite subset $\Lambda \subset Br(X_\eta)$ generating $Br(X_\eta)$ modulo $Br(k(t))$ such that $\Lambda \subset Br(Y)$ where $Y \subset X$ is an open with complement $X \setminus Y$ a disjoint union of closed split fibers.*

Then the Brauer-Manin obstruction is the only obstruction to the Hasse principle and to weak approximation for rational points on X .

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